EXERCISE 1

Given the following system:

$$\dot{x}_1 = x_1 x_2 - 2x_1^2 - (2+p)x_1$$
$$\dot{x}_2 = -x_1 x_2 - p x_2$$

1. compute and classify the equilibria when varying the parameter *p*;
2. plot the trajectories for at least a value of *p;*
3. analyse the possible existence of limit cycles;
4. determine if catastrophic bifurcation may take place and for which values of *p*.

Solution

1. Equilibria

Applyinng the equilibrium condition for continuous time systems $\dot{\mathbf{x}} = 0$, one gets.

$$\bar{x}_1 \bar{x}_2 - 2\bar{x}_1^2 - (2+p)\bar{x}_1 = 0$$
$$-\bar{x}_2(\bar{x}_1 + p) = 0$$

From the second equation we obtain: $\bar{x}_1 = -p$ e $\bar{x}_2 = 0$.

Substituting these expression in the first equation, we determine the three equilibria: $E_1(0;0)$, $E_2(-1-p/2;0)$ e $E_3(-p;2-p)$. For some values of the parameter *p*, these equilibria are partly coincident: if $p = -2$ $E_1 \equiv E_2$ in (0;0); if $p = 2$ $E_2 \equiv E_3$ in (-2;0).

Linearizing the system, we obtain

$$\frac{\partial f}{\partial x} = \begin{bmatrix} x_2 - 4x_1 - 2 - p & x_1 \\ -x_2 & -x_1 - p \end{bmatrix}$$

And evaluating the Jacobian in the 3 equilibria, we have

$$\left[\frac{\partial f}{\partial x}\right]_{E_1} = \begin{bmatrix} -2-p & 0 \\ 0 & -p \end{bmatrix} \qquad \text{eigenvalues: } \begin{matrix} \lambda_1 = -2-p \\ \lambda_2 = -p \end{matrix}$$

$$\left[\frac{\partial f}{\partial x}\right]_{E_2} = \begin{bmatrix} 2+p & -1-p/2 \\ 0 & 1-p/2 \end{bmatrix} \qquad \text{eigenvalues: } \begin{matrix} \lambda_1 = 2+p \\ \lambda_2 = 1-p/2 \end{matrix}$$

$$\left[\frac{\partial f}{\partial x}\right]_{E_3} = \begin{bmatrix} 2p & -p \\ p-2 & 0 \end{bmatrix} \qquad \text{Characteristc polynomial: } \lambda^2 - 2p\lambda + p^2 - 2p = 0$$

which means that the eigenvalues are $\lambda_{1,2} = p \pm \sqrt{2p}$, which are complex conjugate if $p < 0$.

We can now classify the equilibria.

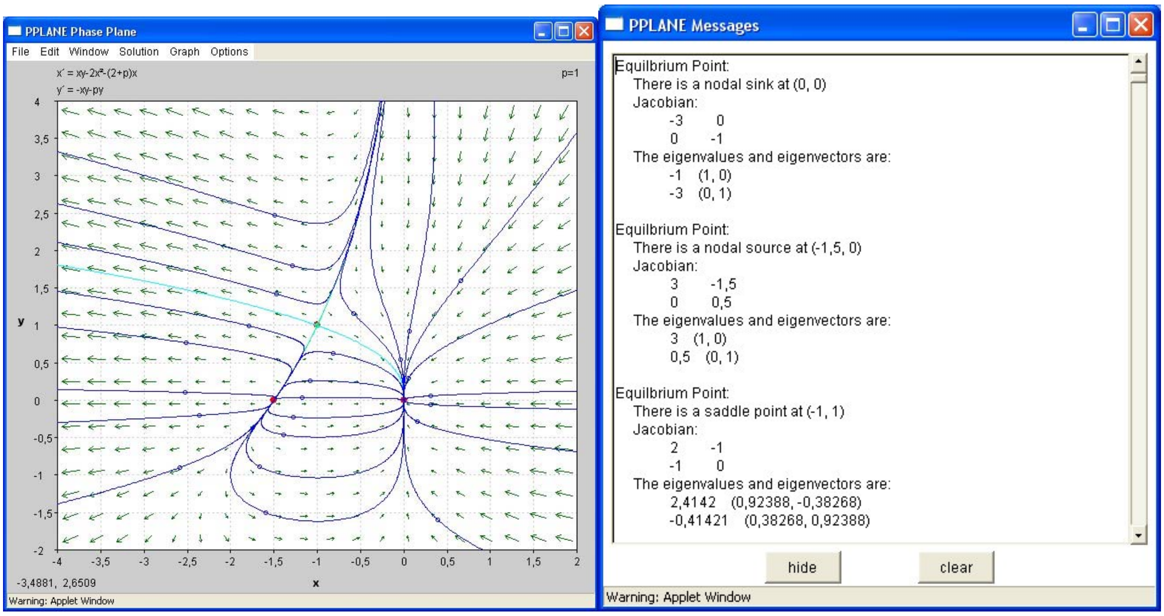
E_1: in a nodal source for $p < -2$; a saddle for $-2 < p < 0$; and a nodal sink for $p > 0$.

E_2: is a saddle for $p < -2$; a nodal source for $-2 < p < 2$; and again a saddle for $p > 2$.

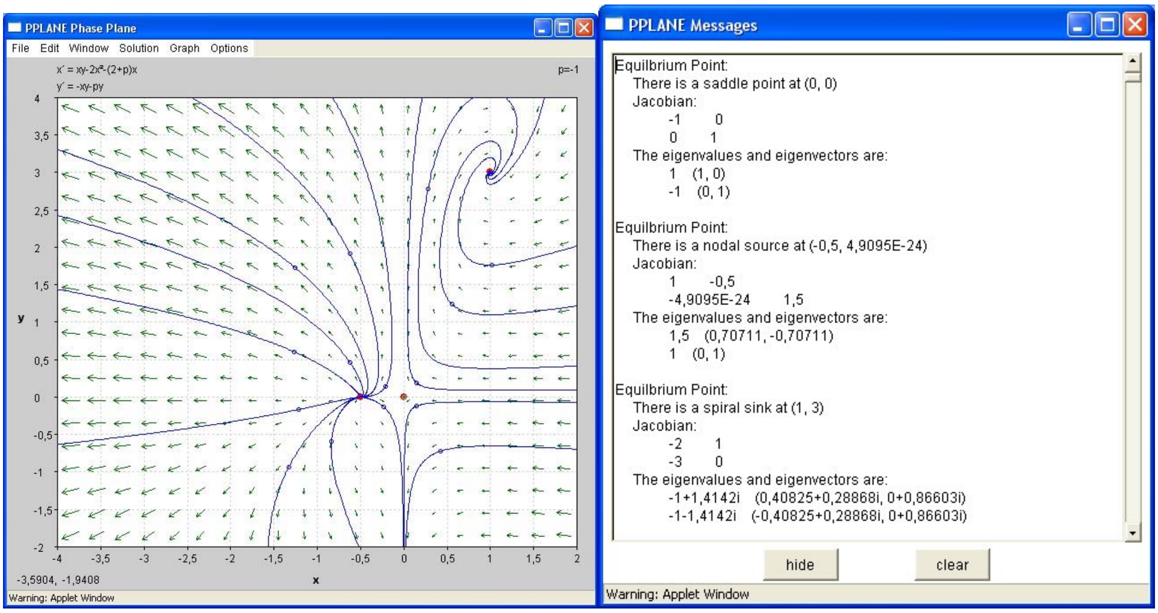
E_3: is a spiral sink for $p < 0$; a saddle for $0 < p < 2$; and a nodal source for $p > 2$.

2. Trajectories

Assuming for instance $p = 1$, one gets:



While, for $p = -1$, the trajectories are:



3. Limit cycles

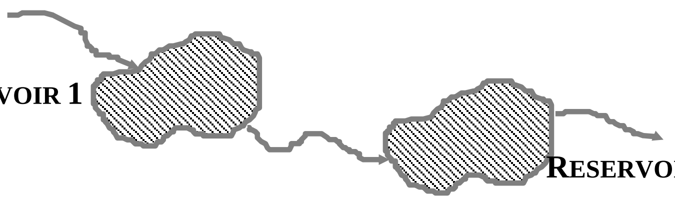
Neither Poincarè theorem nor Bendixon one give a definite answer.
In fact, according to Poincarè theorem limit cycles are possible araound the three equilibria or around single non saddle ones.
Computing the Bendixon function $B = x_2 - 4x_1 - 2 - p - x_1 - p = x_2 - 5x_1 - 2 - 2p$, it turns out that it changes sign along the line $x_2 = 5x_1 + 2(1 + p)$ and thus again cycles intersecting this line may exist.

4. Catastrophic transitions

On the basis of the previous analysis if equilibria, we can conclude that a catastrophic transition does exist for $p = 0$. If the parameter chenges from negative to positive values, equilibrium E_3 changes from spiral sink to saddle and thas the system state has a catastrophic transition toward E_1.

EXERCISE 2

Two reservoirs with surface area S_1 and S_2 are connected in cascade as in the following figure, so that the discharge of the first is the inflow of the second. Both have a dam that regulates the release.



Downstream each of them, there is an agricultural region with a known daily water demand $d_i(t)$, i=1,2, t=1,…,365. The daily inflow values $a(t)$ to the upstream reservoir are also known for a long time period.

REFERRING TO THE ABOVE SYSTEM, define all the interesting variables and answer the following questions:

a) How can the management problem be formulated?
A possibile objective is

$\min \sum_t \left[(d_1(t) - r_1(t))^2 + (d_2(t) - r_2(t))^2 \right]$ *where $r_1(t)$ and $r_2(t)$ are the daily releases of the two reservoirs.*

The summation may be extended to a long horizon (or to infinite) and one may consider only positive difference between the demand and the releases (water deficits).

b) Which variable must be measured?
The reservoir levels $x_1(t)$ and $x_2(t)$ and the inflow $a(t)$.

c) Which are the control variables (decisions)?
The releases $r_1(t)$ and $r_2(t)$.

d) Which are the disturbances (noises) acting on the system?
The rainfall variation which determines a variation of the inflow

e) How can we formulate a feedforward management policy?

$$r_1(t) = f_1(t, a(t))$$
$$r_2(t) = f_2(t, p(t))$$

where $p(t)$ is the toatl rainfall of a certain number of days preceding t.

f) How can a feedback management policy be formulated if there is a single decision maker? And in case of two separate decision makers?

Single decision maker

$$r_1(t) = f_1(t, x_1(t), x_2(t))$$
$$r_2(t) = f_2(t, x_1(t), x_2(t))$$

Two separate decision makers

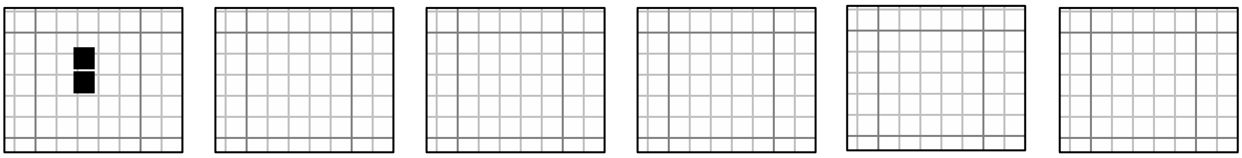
$$r_1(t) = f_1(t, x_1(t))$$
$$r_2(t) = f_2(t, x_2(t))$$

EXERCISE 3

- Define a cellular automaton with binary state 0, 1 and evolution rules different from Conway's (birth with 3 active cells in the neighbor, survival with 2 or 3 active cells, death in all other cases) and at least one equilibrium configuration.

  Example

  RULES: *Von Neuman neighbor; 1 active cell ⇒ survival ; 3 cells ⇒ birth; death in all other cases.*



- To measure the total suspendedn solids the water of a river, turbidity $y(t)$ is measured optically. It has been estimated that it varies daily according to the following law:

  $y(t)$ = 50 + 20 sen(2π t/365) + 10 sen(2π t/[C]) + [N] sen (2π t/3,5),

  where time t is measured in days.

  Which is the minimum sampling rate to follow the avability of turbidity without loosing information?

  Assuming that the gauge has 12 bits to register the information (values between 0 and 4095), which will be the precision of the quatized measure?

According to Shannon theorem, the minimum frequency is at least double of the maximum frequency of the signal, thus the period must be less than 1,75 days (half of the shortest period equal to 3,5 days). In practice, a more reasonable choiche will be 1 or 0,5 days.

Since the range of possible values is between 50+20+10+[N] and 50-20-10-[N], the turbidity will be quantized into 4096 intervals, each of

[50+20+10+[N]-(50-20-10-[N])]/4096 units.

EXERCISE 4

Aswer the following questions:

- What does a "chaotic movement" means?

  It is the movement of a deterministic nonlinear system that remanins in a limited portion of the state space without being periodic.

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- What is a "Hopf bifurcation"?

It is a bifurcation in which the asyptotic behavious shifts from an equilibrium to a cycle (clearly inside the cycle there is still an equilibrium).
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- What is the aim of the “backpropoagation” algorithm in neural networks?

  It is used to fix the weights (parameters) of the network, by propagating backward the errors of the traing set of data.
  ....

- What kind of algorithms one can use to optimise the parameters of a fixed structure management policy?

  All these algorith are iterative and use the parameters of the policy as decision variables. They are grid methods, gradient, Newton,....
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