

## EXERCISE 1

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Given the system:

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -x_2 + 3x_3 + 2u \\ \dot{x}_3 = x_1 + x_2 + x_3 \end{cases}$$
$$y = [N]x_1$$

(1) show that it is unstable; (2) show whether it is possible to stabilize it with a control law of type  $u = \mathbf{k}^T \mathbf{x}$  and (3) with a control law of type  $u = ky$ .

### Solution

The matrixes of the system are:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$
$$\mathbf{c}^T = \begin{bmatrix} [N] & 0 & 0 \end{bmatrix}$$

The first eigenvalue of matrix A is in position (1,1) and it is -1, thus satisfying the condition  $\lambda_i < 0$ . The remaining (2×2) minor has trace 0 e determinant -4. Hence, the system is unstable.

The reachability matrix is as follows:

$$\mathbf{R} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 11 \\ 0 & 3 & 0 \end{bmatrix}$$

and its rank is 3 (the determinant is  $-3(11-2) = -27$ , different from zero).

The system is therefore completely reachable, and the first kind of control law is able to stabilize the system.

As for the second type of control law, it is necessary to substitute the control laws into the system's equation:

$$\begin{cases} \dot{x}_1 = -x_1 + [N]kx_1 \\ \dot{x}_2 = -x_2 + 3x_3 + 4kx_1 \\ \dot{x}_3 = x_1 + x_2 + x_3 \end{cases}$$

Hence, the dynamic matrix of the controlled system is:

$$\tilde{\mathbf{A}} = \begin{bmatrix} -1 + [N]k & 0 & 0 \\ 4k & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

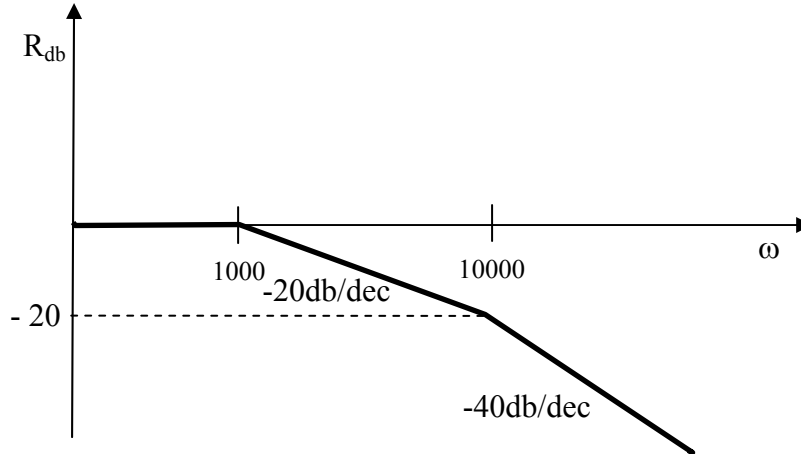
The first eigenvalue is  $-1 + [N]k$ ; hence, in order to be negative (thus making the system stable), the following condition has to be satisfied:  $k < 1/[N]$ . The remaining minor is however unstable, as already shown above; therefore, for no value of k the system can be stabilized. This result is due to  $\mathbf{c}^T$ , which allows for managing the first state variable only via the feedback loop on the output.

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## EXERCISE 2

A noise barrier is being installed along a highway; the frequency response (function of the pulsation  $\omega$ ) of the barrier is shown in the following figure.

Knowing that the vehicles running on the highway emit sound waves mainly with frequency in the range between 1000 and  $[C] \cdot 1000$  Hz (1 Hz = 1 cycle/sec), show which waves will be less attenuated by the barriers, and, even in an approximate way, which is their attenuation.



### Solution

The solution can be found in an exact way, obtaining the transfer function from the Bode diagram, or in an approximate way by simply reading the Bode diagram.

The transfer function is:

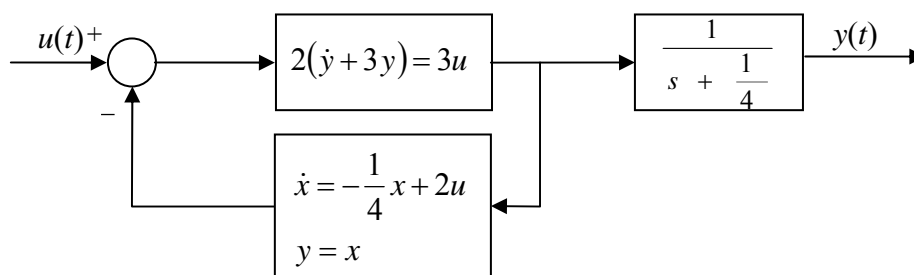
$$G(s) = \frac{1}{(1 + s/1000)(1 + s/10000)}$$

whose absolute value has to be evaluated for  $s = i\omega$ , with  $\omega$  ranging between  $2\pi \cdot 1000$  and  $2\pi \cdot 10000$ . As the system is a low-pass filter, the lowest frequencies will be the least attenuated; it is enough hence to evaluate  $|G(6280i)|$ , which is about 0,13.

Alternatively, one can read from the plot the value corresponding to  $\omega = 6280$ , i.e. about  $-17,5$  dB. One then obtains  $R(6280) = 10^{-17,5/20} = 0,133$ .

## EXERCISE 3

Plot the approximate step response of the system below:



### Solution

The transfer functions of the single blocks are:

$$G_1(s) = \frac{3}{2s + 6} \quad G_2(s) = \frac{1}{1 + 2s} \quad G_3(s) = \frac{2}{s + 1/4}$$

The transfer function of the system as a whole is hence:

$$G(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)} G_3(s) = \frac{3(4s+1)}{(1+2s)(8s^2+26s+30)}$$

The system is asymptotically stable as its poles are:

$$p_1 = -\frac{1}{2}$$

$$p_{2,3} = \frac{-13 \pm \sqrt{169 - 240}}{8}$$

There are hence 2 complex conjugate poles.

As the transform of the step is  $1/s$ , the Laplace transform of the output is:

$$Y(s) = \frac{3(4s+1)}{s(1+2s)(8s^2+26s+30)}$$

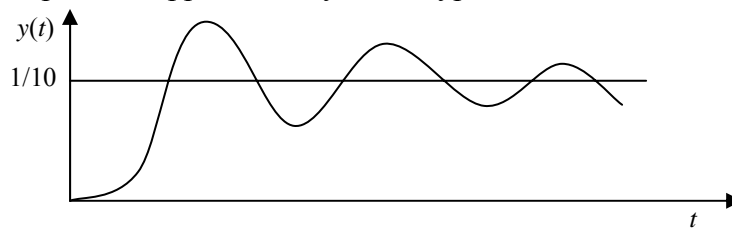
By using the initial and final value theorems, one gets:

$$\lim_{t \rightarrow 0} y(t) = \lim_{s \rightarrow \infty} sY(s) = \lim_{s \rightarrow \infty} \frac{3(4s+1)}{(1+2s)(8s^2+26s+30)} = 0$$

$$\lim_{t \rightarrow 0} \dot{y}(t) = \lim_{s \rightarrow \infty} s^2 Y(s) = \lim_{s \rightarrow \infty} \frac{3s(4s+1)}{(1+2s)(8s^2+26s+30)} = 0$$

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \frac{3(4s+1)}{(1+2s)(8s^2+26s+30)} = \frac{1}{10}$$

Hence, the system response is approximately of the type shown below:



## QUESTIONS

- What is an asymptotic observer?

*It is a dynamic system with the same matrices of the original system, able to asymptotically reconstruct the value of the system state starting from any initial state.*

- For which kind of problems is Kalman filter used?

*It is used for linear problems characterized by random noises on the state and the output.*

- What are the advantages of the Nyquist criterion to analyze the stability of feedback-controlled systems?

*It allows to evaluate the system stability on the base of its blocks, without computing the system's eigenvalues.*

- A linear system has observability matrix  $O = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$ ; is the state  $x' = \begin{bmatrix} 0 \\ [C] \end{bmatrix}$  observable?

*The observability region is a line of type  $[\alpha \quad 3\alpha]^T$ . Hence, the non-observability space, ortogonal to it, is of type  $[\beta \quad -1/3\beta]$ . Hence,  $x'$  does not lie in the non-observability space and therefore it can be observed.*