A simple method to calculate the error probability for 2D RAKE receivers

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Abstract

This paper proposes a simple method to evaluate the average error probability for adaptive antenna array receivers in DS/CDMA systems. Adaptive array can reduce the interference only when the direction of arrivals (DOAs) of the interferers are misaligned (out-beam interferers) with the DOAs of the user of interest. The DOAs let the interferers be partitioned into two equivalence classes according to the beampattern, in-beam and out-beam interferers being counted differently in average error probability evaluation. For propagation over Rayleigh channels the model shows good agreement with simulation results.

1 Introduction

The use of antenna arrays in CDMA systems is motivated by the need to reduce co-channel interference from other users within its own and neighboring cells. In the past capacity improvement with antenna arrays was investigated evaluating the signal to interference-plus-noise or the probability of outage [1], [2]. The exact analytical evaluation of the error probability in CDMA system is still an open subject. Bounds are proposed and "Gaussian approximations" can be found for asynchronous CDMA in AWGN channels [3], [4]. The analytical evaluation of the error probability of a conventional RAKE receiver with adaptive antenna arrays (2D RAKE) needs to take into account, simultaneously, both spatial and temporal diversity. Here we concentrate analyzing spatial diversity as after the beamforming the power of the interferers varies greatly and the Gaussian approximation is inaccurate [3]. The error probability cannot be ascribed to the average interference measured after the spatial filter [5], [2], [6] but it is mostly governed by the worst-case. The error probability formulas for matched filter receivers have to be manipulated in order to take differences into account when the direction of arrivals (DOAs) of the interferers are aligned, or not, with the DOA for the user of interest. These two cases allow a partitioning of the interferers into two spatial equivalence classes depending on beamwidth resolution: the in-beam and the out-beam interferers as shown in fig.1. Each user belonging to one class is counted differently for the evaluation of the error probability.

In-beam/out-beam partitioning of the users needs a simplified model of the angular gain of the spatial filter. In this paper the array beampattern is modelled by a piece-line function that approximates the pass-band (or *in-beam*) with a fixed beamwidth and the stop-band (or *out-beam*) with an equivalent attenuation. The average error probability for K users with DOAs uniformly distributed within a symmetric support around the array broadside has been derived for chip and phase asynchronous CDMA system in AWGN and for propagation over a frequency-flat Rayleigh fading channel (temporal diversity is not considered here).

The paper is organized as follows: the CDMA model and the equivalent beampattern is in Section 2. The approach for the calculation of error probabilities for 2D RAKE receivers is discussed in Section 3 together with validation by simulations.

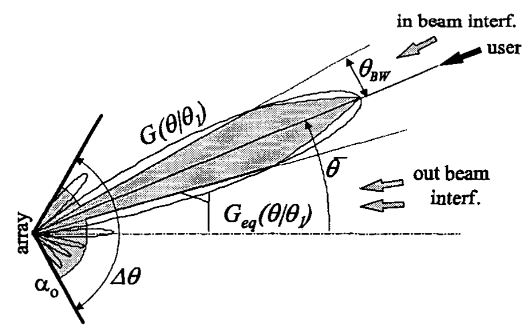

Figure 1. In-beam/out-beam partitioning of the multiple access interference.

2 CDMA system and beamforming

The $M \times 1$ vector of the received signal in a DS-CDMA system with K users employing different normalized signatures $s_1(t), s_2(t), ..., s_K(t)$ and transmitting sequences of binary phase-shift keying (BPSK) symbols can be modeled as

$$\mathbf{r}(t) = \sum_{k=1}^{K} A_k \mathbf{a}(\theta_k) \sum_{l} b_k[l] s_k(t - lT - \tau_k) + \sigma \mathbf{n}(t). \quad (1)$$

For the k-th user: $\mathbf{a}(\theta_k)$ is the $M \times 1$ vector that describes the array response to the DOA θ_k, the m-th component for half-wavelength antenna spacing is $\exp(j(m-1)\pi \sin\theta_k)$ for $1 \leq m \leq M$; A_k is the received complex valued amplitude, $b_k[l] \in \{-1, +1\}$ is the i-th transmitted symbol, τ_k is the relative delay uniformly distributed on $[0, T)$, T is the symbol period (asynchronous model). Noise is white, spatially uncorrelated ($\mathbf{n}(t) \sim \mathcal{N}(0, \mathbf{I})$) and independent of $b_k[l]$. For the sake of simplicity it is assumed that the support of $s_k(t)$ is on the interval $[0, T]$ and that it has unit energy (the symbol energy is $|A_k|^2$). After the beamforming with the spatial filter $\mathbf{w}_\ell$ the output of the ℓ-th filter matched to $s_\ell(t)$ is

$$\begin{aligned} y_\ell[i] &= \int s_\ell(t - iT - \tau_\ell) \mathbf{w}_\ell^H \mathbf{r}(t) dt = \\ &= \sum_{k=1}^{K} A_k \sum_{l} b_k[l] G(\theta_k|\theta_\ell) \rho_{\ell k}(\tau_\ell - \tau_k) + \sigma n'_\ell(i) \quad (2) \end{aligned}$$

where $n'_\ell(t) \sim \mathcal{N}(0, 1/M)$ as $\mathbf{w}_\ell^H \mathbf{w}_\ell = 1/M$, $G(\theta_k|\theta_\ell)$ is the spatial gain of the uniform linear array beamformer designed for the specified angle θ_ℓ so that $|G(\theta_k|\theta_\ell)| \leq |G(\theta_\ell|\theta_\ell)| = 1$, $\rho_{\ell k}(\tau_\ell - \tau_k) = \int s_\ell(t - iT - \tau_\ell) s_k(t - lT - \tau_k) dt$ is the asynchronous crosscorrelation between signatures [11]. In this paper we limit the analysis to the case of users with independent and uniformly distributed DOAs: $\theta_k \sim \mathcal{U}(-\Delta\theta/2, +\Delta\theta/2)$. Amplitudes can be either deterministic $|A_k| = A$ for $\forall k$ or circularly Gaussian zero mean random variables, and $E[|A_k|^2] = 2\sigma_A^2$ (frequency-flat Rayleigh fading). All the users have the same average power as in a system with perfect power control.

For the user of interest (say θ_1) conventional beamforming weights are considered $\mathbf{w}_1 = \mathbf{a}(\theta_1)/M$, the angular gain function is $G(\theta|\theta_1) = \mathbf{a}(\theta_1)^H \mathbf{a}(\theta)/M$. To simplify the computations in the following, the gain $|G(\theta|\theta_1)|^2$ can be approximated by a piece-line function $|G_{eq}(\theta|\theta_1)|^2$ that models the pass-band (or *in-beam* with support $\Theta(\theta_1) = [\theta_1 - \theta_{BW}(\theta_1), \theta_1 + \theta_{BW}(\theta_1)]$) with a linear gain and the stop-band (or *out-beam* with support $\bar{\Theta}(\theta_1)$) with an equivalent attenuation α_o (see fig.1):

$$|G_{eq}(\theta|\theta_1)|^2 = \begin{cases} 1 - \frac{1}{2}\frac{|\theta - \theta_1|}{\theta_{BW}(\theta_1)} & \text{for } \theta \in \Theta(\theta_1) \\ \alpha_o & \text{for } \theta \in \bar{\Theta}(\theta_1) \end{cases} \quad (3)$$

The beamwidth $2\theta_{BW}(\theta_1)$ depends on the number of antennas M and on θ_1. The support $\Theta = \Theta(\theta_1) \cup \bar{\Theta}(\theta_1) = [-\Delta\theta/2, \Delta\theta/2]$ covers all the admissible DOAs (e.g., in a mobile system with 3-cell sectorization the angles range within ± 60deg). The parameters of $G_{eq}(\theta|\theta_1)$ can be found numerically by transforming the optimization in the wavenumbers $\omega = \pi \sin\theta$ for $\theta_1 = 0$ as in this case the sin(.) stretching can be neglected:

$$\int_{\Omega} \left(\frac{\sin \omega M/2}{M \sin \omega/2} \right)^2 d\omega = \frac{3}{2}\omega_{BW} \quad (4a)$$

$$\int_{\bar{\Omega}} \left(\frac{\sin \omega M/2}{M \sin \omega/2} \right)^2 d\omega = \alpha_o(\Delta\omega - 2\omega_{BW}) \quad (4b)$$

the supports are $\Omega = [-\omega_{BW}, \omega_{BW}]$ and $\Omega \cup \bar{\Omega} = [-\Delta\omega/2, \Delta\omega/2]$, $\omega_{BW} = \pi \sin\theta_{BW}$ and $\Delta\omega/2 = \pi \sin \Delta\theta/2$. The optimized values (θ_{BW}, α_o) carried out for varying the number of antennas M are shown in fig.2. Once $\bar{\theta}_{BW}$ has been optimized for a specified value of $\bar{M}$ the beamwidth can be re-scaled to any value of M according to the relationship $\theta_{BW} = (M/\bar{M})\bar{\theta}_{BW}$ as shown in fig.2-top (solid line). For small deviations from the broadside the beamwidth θ_{BW} optimized for $\theta_1 = 0$ is stretched as $\theta_{BW}(\theta_1) = \theta_{BW}/\cos\theta_1$ [7].

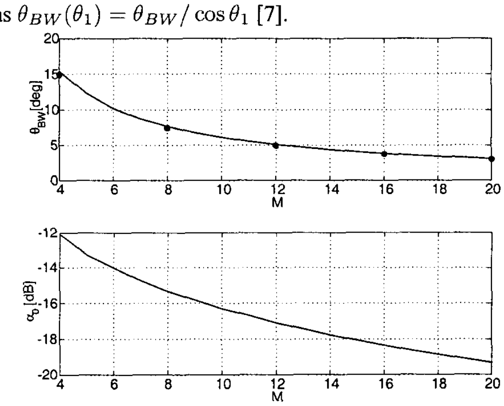

Figure 2. Equivalent beamforming parameters: (θ_{BW}, α_0) **vs.** M.

3 Error probability with adaptive array

In the CDMA system with adaptive array of antennas the effect of spatial filter is to enhance the differences in the power of the interfering users, and system performance is mainly governed by the worst case that occurs when the interferers are angularly aligned with the user of interest. The interferers are partitioned into two interference-driven spatial equivalence classes: the in-beam and the out-beam interferers depending on whether the users have in-beam and out-beam DOAs with respect to the user of interest (fig.1). For each user of interest characterized by the DOA θ_1, the remaining $K-1$ users are partitioned into the two disjointed subsets $\{2,3,...,K\} = B(\theta_1) \cup \bar{B}(\theta_1)$ such that $\theta_j \in \Theta(\theta_1)$ if $j \in B(\theta_1)$. Within each spatial equivalence class the users have (approximately) the same power, approximation for bit error performance can be considered for this case. The in-beam and the out-beam users contribute to the level of interference according to the cardinality of each set, $|B(\theta_1)|$ and $|\bar{B}(\theta_1)|$ respectively. The $|B(\theta_1)|$ in-beam users contribute to the overall level of interference at the decision variable, the remaining $|\bar{B}(\theta_1)| = K-1-|B(\theta_1)|$ asynchronous users can be assimilated to Gaussian noise and thus contribute to modifing the decision variable. In this latter case the AWGN can be increased according to the "standard Gaussian approximation" (see e.g., [3]) $\sigma_I^2 = \alpha_o A^2 |\bar{B}(\theta_1)|/(3N)$.

Let $P_e(A^2, \sigma^2, K_I)$ be the error probability for asynchronous CDMA with one antenna system ($M=1$) that depends on the noise power σ^2 and the overall number of interferers K_I, the error probability for adaptive antenna system with $|B(\theta_1)|$ in-beam interferers uniformly distributed within $\Theta(\theta_1)$ for a specific DOA θ_1 can be reduced to

$$P(E \mid |B(\theta_1)|, \theta_1) = P_e(A^2, \frac{\sigma^2}{M} + \frac{\alpha_o A^2 |\bar{B}(\theta_1)|}{3N}, \frac{3}{4}|B(\theta_1)|). \tag{5}$$

The array gain $1/M$ with respect to the AWGN has already been included while the correction term (3/4) on the cardinality $|B(\theta_1)|$ depends on the average energy of $|B(\theta_1)|$ in-beam interferers being slightly attenuated and uniformly distributed in $[A^2/2, A^2]$, see ref.[8]. The average bit-error-rate for an adaptive array of antennas with random DOAs is obtained by taking the expectation with respect to $|B(\theta_1)|$ and θ_1

$$P(E) = \mathrm{E}\{\mathrm{E}\{P(E||B(\theta_1)|, \theta_1)\}\}. \tag{6}$$

For $\theta_k \sim \mathcal{U}(-\Delta\theta/2, +\Delta\theta/2)$ the probability of $K_I \in [0, K-1]$ in-beam interferers is

$$p(|B(\theta_1)|) = p(K_I|\theta_1) = \binom{K-1}{K_I} \lambda^{K_I} (1-\lambda)^{K-K_I-1}. \tag{7}$$

It depends on the ratio $\eta := 2\theta_{BW}/\Delta\theta$ as $\lambda = \lambda(\theta_1) = \eta/\cos\theta_1$ is the probability to have one in-beam interferer for uniformly distributed DOAs. The average error probability (6) reduces to [8]

$$P(E) \simeq \sum_{K_I=0}^{K-1} \mathcal{X}\, \eta^{K_I} \binom{K-1}{K_I} P_e(A^2, \frac{\sigma^2}{M} + \sigma_I^2, \frac{3K_I}{4}) \tag{8}$$

where $\sigma_I^2 = \alpha_o A^2 (K-1-K_I)/(3N)$. $\mathcal{X}$ depends on the beam stretching $1/\cos\theta_1$ in $\lambda(\theta_1)$, in practice when $\Delta\theta < 60 \div 70$ deg it can be approximated as $\mathcal{X} \simeq (1-\eta)^{K-K_I-1}$.

The performance (8) can be adapted to different receivers depending on the probability of error of the scalar receiver $P_e(A^2, \sigma^2, K_I)$ as discussed below.

Remark 1: The DOAs of all the users are assumed known and spatial filter is based on conventional beamforming. In practice, the results shown here remain valid even if the DOAs are estimated with an error that is much smaller than the beamwidth $\theta_{BW}(\theta_1)$.

Remark 2: The error probability depends mainly on the probability of in-beam users. Beamforming methods can be optimized to reduce out-beam interference [7] (and thus reduce α_o) but they unavoidably preserve the power of in-beam users. Therefore the calculation of the bit error probability can be based on the approximation (8) and on the beamforming parameters in fig.2 even if the beamforming employed differs from the conventional beamforming.

3.1 2D RAKE receiver - no fading

In a multipath environment, the 2D RAKE receiver employs the maximal ratio combining by filtering each path of each user's signal with the corresponding space filter, and then combining the matched filter outputs [9]. The interference rejection of the 2D RAKE receiver relies on the correlation properties of the spreading signatures and on the beamforming of the adaptive antenna arrays, these two aspects will first be considered for propagation over non-fading channel. The average error probability for a chip and phase asynchronous CDMA with single-antenna has been widely investigated in the past. A simple but accurate approximation was derived under the Gaussian approximation for random spreading sequences of length N [3]

$$P_e(A^2, \sigma^2, K_I) = Q\left[\left(\frac{\sigma^2}{A^2} + \frac{K_I}{3N}\right)^{-1/2}\right], \tag{9}$$

$Q[.]$ is the Q-function (in a RAKE receiver with $M=1$ the number of interferers are $K_I = K-1$). The error probability (9) substituted in the approximation with adaptive array (8) gives the approximate relationship for the error probability for an adaptive antenna array receiver, based on the in-beam/out-beam partitioning of the interferers

$$P(E) \simeq \sum_{K_I=0}^{K-1} \mathcal{X}\, \eta^{K_I} \binom{K-1}{K_I} Q\left[\left(\frac{\sigma^2}{MA^2} + \gamma(K_I)\right)^{-1/2}\right] \tag{10}$$

where

$$\gamma(K_I) = \frac{K_I}{4N} + \frac{\alpha_o(K-1-K_I)}{3N} \tag{11}$$

denotes the equivalent noise increasing due to the multiaccess interference.

Simulations for phase and chip asynchronous CDMA have been carried out to validate the model (10) with respect to those approximations that estimate the probability

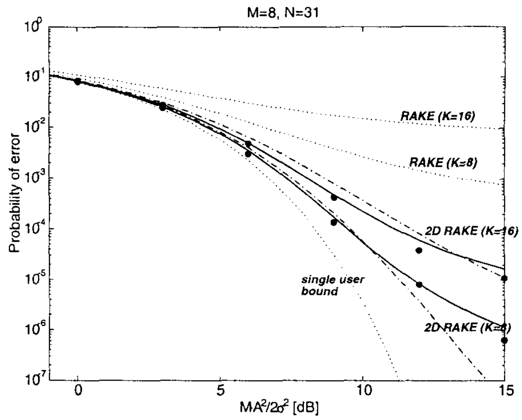

Figure 3. Error probability of 2D RAKE receiver vs. SNR $MA^2/2\sigma^2$.

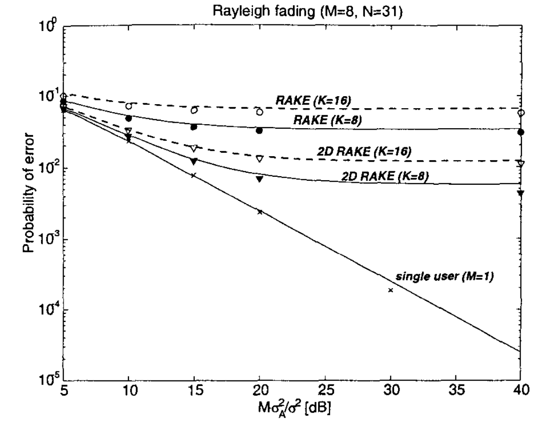

Figure 4. Error probability of 2D RAKE receiver with Rayleigh fading channel vs. SNR $M\sigma_A^2/\sigma^2$.

of error based on the average level of inteference, see e.g., [2], [5]. Here we consider the case of $M = 8$ antennas, random spreading sequence $N = 31$, rectangular pulse shaping, $K = 8$ and 16 users. In fig.3 the signal to noise ratio (SNR) has been scaled with respect to the array gain $1/M$, thus highlighting the spatial diversity gain with respect to the array gain $1/M$. The approximation of Song and Kwon [5] (dash-dot line) under-estimates the probability of error for large SNR while $P(E)$ in eq.(10) (solid line) hugs the simulated results (marks). In addition, the spatial diversity gain with respect to the RAKE ($M = 1$) receiver (dotted lines) demonstrates the efficacy of adaptive array systems in reducing the interference level. The results shown here demonstrate that the key parameters in 2D RAKE receivers is the ratio $\eta = 2\theta_{BW}/\Delta\theta$, that accounts for the probability of having the in-beam interferer when the users are uniformly distributed in $\Delta\theta$. In a mobile system with the base station at the corner of the hexagonal cell the users can be assumed to be uniformly distributed within the cell but the DOAs are far from being uniformly distributed. In this case the model (10) is quite optimistic and a slight correction is needed.

3.2 2D RAKE receiver - Rayleigh fading

The error probability $P_e(\sigma_A^2, \sigma^2/M, K_I)$ for a flat-fading channel (without the distortion of the signature waveform and self-noise interference) can be derived herein by extending the results in ref.[11]. The error probability of BPSK for propagation over Rayleigh faded channel (i.e., $K = 1$ and $M = 1$) is known to be [10]

$$P_e = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\sigma^2/\sigma_A^2}}\right). \qquad (12)$$

In the CDMA system the overall impairment is the superposition of the AWGN and the K_I interferers. The error probability for asynchronous CDMA with K_I interferers is equivalent to the error probability for two fictitious interferers per actual interferer [11] thus doubling the overall number of interferers. For random spreading sequence of length N and rectangular pulse shaping the power of the interference it can be shown that $\sigma_I^2 = 2\sigma_A^2 K_I/3N$. The probability of error of the asynchronous CDMA system for Rayleigh fading channel and one antenna becomes

$$P_e(\sigma_A^2, \sigma^2, K_I) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\frac{\sigma^2}{\sigma_A^2}+\frac{2K_I}{3N}}}\right), \qquad (13)$$

recall that for $M = 1$ then $K_I = K - 1$. The error probability for an adaptive array of M antennas can be obtained by substituting (13) in (8) and increasing the overall noise to signal ratio accordingly:

$$P(E) \simeq \sum_{K_I=0}^{K-1} \mathcal{X}\eta^{K_I}\binom{K-1}{K_I}\frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\frac{\sigma^2}{M\sigma_A^2}+\gamma(K_I)}}\right), \qquad (14)$$

here $\gamma(K_I) = K_I/2N + 2\alpha_o(K-1-K_I)/3N$. It can be noticed that the numerical computation (14) is simple once an evaluation is made of the terms that depend on M from the approximate spatial filter (fig.2).

Fig.4 shows the error probability for 2D RAKE receivers with adaptive antenna arrays and flat-fading (no time-diversity is considered, see the remark below) under the same conditions as fig.3 ($M = 8$, $N = 31$) for $K = 8$ (solid line) and $K = 16$ (dashed line) users. The error probability for the RAKE receiver (13) and for the 2D RAKE receiver (14) show close correspondence with the simulations (marks). It can be shown that by increasing the number of antennas in the array, performance reaches that of single-user in the AWGN (12) (solid line), the probability for in-beam interference being reduced by decreasing the ratio $\eta = 2\theta_{BW}/\Delta\theta$.

The error probability vs. increasing the number of users K and $M = 8$ (solid line) or $M = 16$ (dashed line) is in fig.5 for both Rayleigh fading (SNR is $MA^2/2\sigma^2 = 6dB$) and no-fading (SNR is $M\sigma_A^2/\sigma^2 = 20dB$) channel. Again there is good agreement between the proposed, simplified, model and the simulations (marks). As a rule of thumb, given an SNR, the probability of error depends on the ratio K/M.

Remark 3: The proposed model can be extended to the case of the 2D RAKE receiver for propagation over frequency-selective fading channels. In this case the order of the time diversity L for all the users must be taken into account in a similar fashion as proposed in ref.[10]. It should be noted that the spatial diversity depends on the probability for in-beam interference $\lambda(\theta_1)$ as for the flat-fading channels, what has to be changed is the maximum number of virtual interferers (up to $L(K-1)$ for the all the interfererers) and the error probability (13).

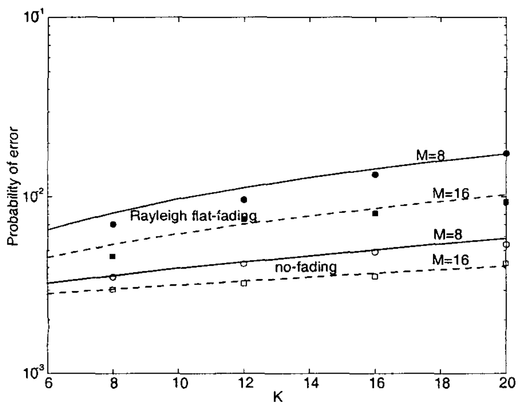

Figure 5. Error probability of 2D RAKE receiver vs. the number of users (K)

4 Conclusions

The method proposed in this paper for the analysis of the average error probability in a CDMA system is based on the angular partitioning of the users, according to the shape of the beampattern. The partition of the users into in-beam and out-beam interferers greatly simplifies the analysis for a 2D RAKE receiver with adaptive beamforming. This simple model shows good agreement with simulation results, and can be extended to the case of space-time multiuser detection as proposed in ref.[8]. The analytical model can be part of a simulation package where average performance can be evaluated with reduced simulation runtimes only by the analysis of the geometrical locations of the mobile terminals.

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