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2-D Phase Unwrapping

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SUMMARY

This paper presents a method to unwrap 2D phases. Unwrapped phase can be considered as the superposition of irrotational and rotational phase fields. The irrotational field is determined by solving a Laplace problem with Neumann boundary conditions. This solution is unambiguous but for an arbitrary constant phase shift; the unwrapped irrotational component is then wrapped and subtracted from the original field. The residual isolates the ambiguous phase field.
This approach has the advantage that a unique unwrapped phase component is determined plus having isolated one of the solutions of the ambiguous phase field. This procedure is more robust than unwrapping by line integrals because it does not propagate errors. Aliasing effects in the 2D domain are also addressed.

As a real data example the phases of refracted first arrivals are unwrapped in the source-receiver domain. The weathering and the refractor phase distortion are estimated.

INTRODUCTION

Phase information is fundamental in signal processing (A.V. Oppenheim and J.L. Lim 1981). When phase is measured modulo-2π it is called principal value (p.v.). To be useful for linear processing, phase has to be unwrapped: a continuous function is recovered and the 2π discontinuities are hopefully eliminated. In 1D domains, phase unwrapping algorithms integrate the phase difference between two contiguous points (J.M.Tribolet 1977). Extending 1D algorithms to 2D phase fields does not give satisfactory results, unless singular points and the connecting branch cuts are identified (C.Prati et al. 1990).
As an alternative, H.Takajo and T.Takahashi (1988) consider the phase as a potential field sum of rotational and irrotational components; they unwrap phase from the irrotational (curl-free) component.

This paper presents a technique for 2D unwrapping following the H.Takajo and T.Takahashi formalism. The irrotational phase field is unwrapped using a relaxation technique with Neumann boundary conditions. The p.v. phase differences are imposed on the boundaries. The ambiguous rotational phase field is obtained from the wrapped residual (i.e. the difference between wrapped phase and p.v. irrotational phase field) .

The present 2D phase unwrapping technique has several fields of applications such as surface-consistent deconvolution, Phase vs. Offset processing of prestack data, Rytov approximation tomography (M.J.Woodward 1989) and complex cepstrum processing (Oppenheim et al. 1968). As an example the phase of refracted first arrivals is unwrapped in the source-receiver domain (s,r) leading to a surface-consistent phase measurement.

PHASE ALIASING

In the complex variable $a(x)\exp\{j\theta(x)\}$, we indicate with $[\theta(x)]_p$ the principal value (p.v.) of the phase; all phases that differ from the p.v. by multiples of 2π contain the same information. In the following $[\,.\,]_p$ indicates the p.v. wrapping operator while the index $_w$ indicates the wrapped phase to be unwrapped. Obviously the wrapped phase should be the p.v. of the unwrapped phase (congruency condition).
There is ambiguity also in phase difference estimation. When the information of interest is the exponential $\exp\{j(\theta_2-\theta_1)\}$, the phase difference of the wrapped phase will be defined from the p.v. of the phase difference as $\Delta\theta_{12_{n,m}} = [\theta_2-\theta_1]_p + 2(n-m)\pi$, $(n,m = 0,\pm1,\pm2,\ldots)$. Unwrapped phase information is contained into the unknown quantities $2\pi(n-m)$ of the measured p.v. phase difference.

The 1D wrapped phase $\theta_w(x)$, or p.v. phase, is obtained from the unwrapped phase $\theta(x)$ as follow

$$\theta_w(x) = \theta(x) + 2\pi\sum_i a_i step(x-x_i); \tag{1}$$

x_i are the discontinuities where unwrapped phase differs from wrapped phase by multiples of 2π; the step function (*step*) has amplitude $a_i = \pm1, \pm2, \ldots$. The first derivative of (1) becomes

$$\frac{d\theta_w(x)}{dx} = \frac{d\theta(x)}{dx} + 2\pi\sum_i a_i\delta(x-x_i). \tag{2}$$

The 1D phase unwrapping identifies the sequence of impulses $\sum_i a_i\delta(x-x_i)$ from the knowledge of wrapped phase. The aliasing conditions for a sampled phase in 1D domain are: 1.) the step amplitude should be $a_i = \pm1$; 2.) the distance between 2 discontinuities should be $Min_{\,i\neq j} \mid x_i - x_j \mid \geq 2\,\Delta x$ where Δx is the sampling interval. These conditions require that the effective maximum phase variation between two samples should be less that $\pm\pi$.
On the other hand, incorrect estimation of the impulses from wrapped phase leads to the propagation of the phase unwrapping error.

Let us now consider the phase $\theta(x,y)$ defined in a 2D domain Ω; the wrapped phase $\theta_w(x,y)$ is obtained from the unwrapped phase by a combination of 2D steps where each step lies along a line of equation $\alpha_i(x,y) = 0$ in Ω. Let us indicate with $\vec{P}(x,y)$ the vector identifying the point (x,y); then $\vec{P}_i(x,y)$ is the coordinate-vector along the line $\alpha_i(x,y) = 0$. The wrapped phase becomes:

$$\theta_w(x,y) = \theta(x,y) + 2\pi \sum_i a_i step_{2D}(\vec{P}(x,y) - \vec{P}_i(x,y)), \quad (3)$$

where $step_{2D}(\vec{P}(x,y))$ is a 2D step function. Applying the gradient to (3) follows

$$\vec{\nabla}\theta_w(x,y) = \vec{\nabla}\theta(x,y) + 2\pi \sum_i a_i \vec{\delta}(\vec{P}(x,y) - \vec{P}_i(x,y)), \quad (4)$$

the vector $\vec{\delta}(\vec{P}(x,y) - \vec{P}_i(x,y))$ describes, along the line $\alpha_i(x,y) = 0$, a front of oriented impulses turned towards the increasing 2D step function. The aliasing conditions depend on the implicit lines $\alpha_i(x,y) = 0$ in Ω that should be recovered from the wrapped phase:

- $\alpha_i(x,y) = 0$ is a continuous line in the domain Ω and can become, as a limit, an isolated point.
- The line $\alpha_i(x,y) = 0$ can be either a closed line or it should have the extreme points coincident with the boundaries of the domain Ω.

The extreme points of line $\alpha_i(x,y) = 0$ are called *singular points.* The detection of singular points can be achieved from wrapped phase. They make the phase unwrapping ambiguous and incorrectly unwrapping error propagation arises.

2D PHASE UNWRAPPING

Let us consider the phase difference of the wrapped phases in a sampled domain Ω (it has been defined: $x = x_i = i\Delta x$ and $y = y_j = j\Delta y$):

$$[\Delta\theta_x(x_i,y_j)]_p = [\theta(x_{i+1},y_j) - \theta(x_i,y_j)]_p = [\Delta\theta_i(i,j)]_p \quad (5)$$

$$[\Delta\theta_y(x_i,y_j)]_p = [\theta(x_i,y_{j+1}) - \theta(x_i,y_j)]_p = [\Delta\theta_j(i,j)]_p. \quad (6)$$

The *principal value phase gradient* $\vec{\nabla}_p\theta(x,y)$ for a wrapped phase $\theta_w(x,y)$ is defined as

$$\vec{\nabla}_p\theta_w(x,y) = [\Delta\theta_i(i,j)]_p\,\vec{\imath} + [\Delta\theta_j(i,j)]_p\,\vec{\jmath}, \quad (7)$$

where $\vec{\imath}$ and $\vec{\jmath}$ are unit vectors in x and y directions. Whenever the 2D phase has not been aliased, the p.v. of the phase gradient for the wrapped phase is equivalent to the unwrapped phase gradient. In other words, the p.v. of the phase difference measured for the wrapped phase is coincident with the unwrapped phase difference. Under this condition, the p.v. of the phase gradient is uniquely determined from the gradient of a scalar function which is the unwrapped phase; the phase field that is generated from the p.v. of the phase gradient is irrotational.

In general, the phase field generated by the p.v. of the phase gradient is defined in the domain Ω:

$$\vec{\nabla}_p\theta_w(x,y) = \vec{\Psi}(x,y). \quad (8)$$

The unwrapped phase $\theta(x,y)$ should satisfy the congruency conditions:

$$\theta_w(x,y) = [\theta(x,y)]_p, \quad (9)$$

$$\vec{\nabla}_p\theta(x,y) = \vec{\nabla}_p\theta_w(x,y). \quad (10)$$

The phase field can be decomposed into an irrotational phase field ($\vec{\Psi}_I(x,y)$) and into a rotational phase field ($\vec{\Psi}_R(x,y)$):

$$\vec{\Psi} = \vec{\Psi}_I + \vec{\Psi}_R \quad (11)$$

where $\vec{\nabla} \wedge \vec{\Psi}_I = 0$ and $\vec{\nabla} \times \vec{\Psi}_R = 0$ hold. From the analysis of (11) the condition for unambiguous phase unwrapping follows. From the divergence of (11) follows that p.v. of the Laplace operator for the wrapped phase

$$\begin{aligned}\nabla_p^2\theta_w(i,j) \;=\; & [\Delta\theta_i(i-1,j)]_p - [\Delta\theta_i(i,j)]_p + \\ & [\Delta\theta_j(i,j-1)]_p - [\Delta\theta_j(i,j)]_p, \end{aligned} \quad (12)$$

can be used for phase unwrapping the phase of the irrotational component $\theta_I(x,y)$:

$$\vec{\nabla} \times \vec{\Psi} = \nabla_p^2\theta_w(x,y) = \nabla^2\theta_I(x,y). \quad (13)$$

The integration of (13) requires boundary conditions. Unless the unwrapped phase has an a-priori condition, in phase unwrapping the p.v. of the normal derivative of the wrapped phase at the boundary are examined (Neumann condition). Generally, irrotational unwrapped phase with Neumann condition has a constant phase shift that is of scarce importance in phase unwrapping. Nevertheless the phase shift can be determined minimizing the difference between complex variables. The relation (13) is integrated using relaxation techniques for electrical and magnetic field problems (K.J.Binns and P.J.Lawerson 1973).

The rotational phase field has many solutions. The rotational component is generated from points in the domain Ω that correspond to the singular points of the lines $\alpha_i(x,y) = 0$ previously considered. The integration of the rotational phase field along an arbitrary closed path in Ω that contain singular points is

$$\oint \vec{\Psi}_R \times d\vec{l} = \sum_i 2\pi n_i \quad (14)$$

where n_i is the multiplicity of each singular point. The (14) underlines the ambiguity of phase unwrapping when singular points are present and the same relation is useful for singular points detection (others phase unwrapping algorithms try to connect singular points through application-driven strategies (C.Prati et al. 1990)).
The phase unwrapping of the rotational component considered follows from the congruency condition

$$\theta_R(i,j) = [\theta_w(i,j) - [\theta_I(i,j)]_p]_p. \qquad (15)$$

The constant phase shift of the irrotational phase component makes the rotational component ambiguous. Depending on the choice of the phase shift, the rotational component contains 2π discontinuities along arbitrary lines connecting singular points.
The unwrapped phase $\theta(x,y) = \theta_I(x,y) + \theta_R(x,y)$ is a combination of a unique phase field (irrotational) and an ambiguous phase field (rotational) that depends on the choice of the phase shift.
In Fig.1 (a.) is presented an example of 2D unwrapped phase that has a 2π step. The algorithm has been applied to the p.v. phase. The irrotational unwrapped phase (Fig.1 (b.)) is smooth and recovers the unambiguous phase component; the rotational unwrapped phase (Fig.1 (c.)) has the 2π discontinuity. The unwrapped phase (Fig.1 (d.)) has wrongly (due to the ambiguous rotational phase component) placed the 2π discontinuity connecting the two singular points. Even noisy (gaussian noise with $\sigma_{noise} \simeq \pi/6$) wrapped phase can be recovered successfully with the discontinuity placed in the same position (Fig.1 (e.)).

AN APPLICATION: Surface-Consistent Phase Measurements from Refracted First Arrivals

The refracted first arrivals phase measurement (U.Spagnolini 1990) $\theta(r,s)$ from the source s and the receiver r coordinates will be considered as a sum of three terms:

$$\theta(r,s) = \hat{\theta}(s) + \int_s^r \hat{k}(x)\,dx + \hat{\theta}(r), \qquad (16)$$

where $\hat{\theta}(s)$ and $\hat{\theta}(r)$ are the source and the receiver phase shift; $\hat{k}(x)$ is the refractor specific phase shift. The decomposition is the same that is usually performed for the computation of refracted first arrivals times. Phase measurements require 2D phase unwrapping before equation (16) fitting is performed. In Fig.2 is shown the unwrapped phase of the refracted first arrivals using line integration (a.) and the phase field decomposition (b.). The branch-cut strategy for connecting singular points leaves several 2π discontinuity while phase field decomposition enhances the common receiver parallelism (E.Brückl 1987).
After phase unwrapping and model fitting, source and receiver phase shift is estimated together with the refractor specific phase shift. The source and receiver phase shift appears proportional to the weathering thickness, the estimated weathering specific phase shift ($\simeq 10^{-2}\,rad/m$) is less than the refractor specific phase shift ($\simeq 3.5\,10^{-3}\,rad/m$). This corresponds to different phase distorsion in the two media (G.P. Angeleri and E. Loinger 1984).

CONCLUSIONS

A 2D phase unwrapping algorithm is examined. Aliasing and noise make the phase unwrapping ambiguous. Unwrapped aliased phase is always affected by errors: a careful implementation can limit error propagation and build-up. If a-priori conditions are available, even aliased phase can be properly unwrapped.
Ambiguous unwrapping is due to a rotational phase component whose sources are the singular points. The unambiguous irrotational phase field is obtained by relaxation with Neumann boundary conditions. If the unwrapped phase is known on the boundary, Dirichlet conditions lead to a more robust algorithm. This may be the case with particular applications. The algorithm examined in this paper is less affected by errors propagation than the algorithms based upon line integration.

A real data example of phase unwrapping is given with measurements of surface-consistent phase. There is considerable room for further research and understanding.

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REFERENCES

G.P.Angeleri and E.Loinger 1984, *Phase distortion due to absorption in seismograms and VSP*, Geophysical Prospecting 32, p.406-424.

K.J.Binns and P.J.Lawrenson 1973, *Analysis and Computation of Electric and Magnetic Field Problems*, Pergamon Press.

E.Brückl 1987, *The interpretation of traveltime fields in refraction seismology*, Geophysical Prospecting 35, p.973-992.

A.V.Oppenheim and J.S.Lim 1981, *The Importance of Phase in Signals*, Proc.IEEE vol.69, p.529-541.

A.V.Oppenheim, R.W.Schafer and T.G.Stockham 1968, *Nonlinear filtering of multiplied and convolved signals*, Proc. IEEE vol.65, p.1264-1291.

C.Prati, F.Rocca, A.M.Guarnieri and E.Damonti 1990, *Seismic migration for SAR focusing: interferometrical applications*, IEEE Trans. Geosci. Remote Sens., vol.28 n.4, p.627-640.

U.Spagnolini 1991, *Adaptive picking of refracted first arrivals*, Geophysical Prospecting 39, p.293-312.

J.M.Tribolet 1977, *A new phase unwrapping algorithm*, IEEE Trans. on ASSP, vol.25 n.2, p.170-177.

H.Takajo and T.Takahashi 1988, *Least-squares phase estimation from the phase difference*, J.Opt.Soc.Am. vol.5 p.416-425.

M.J.Woodward 1989, *Wave equation tomography*: Ph.D. thesis, Stanford University.

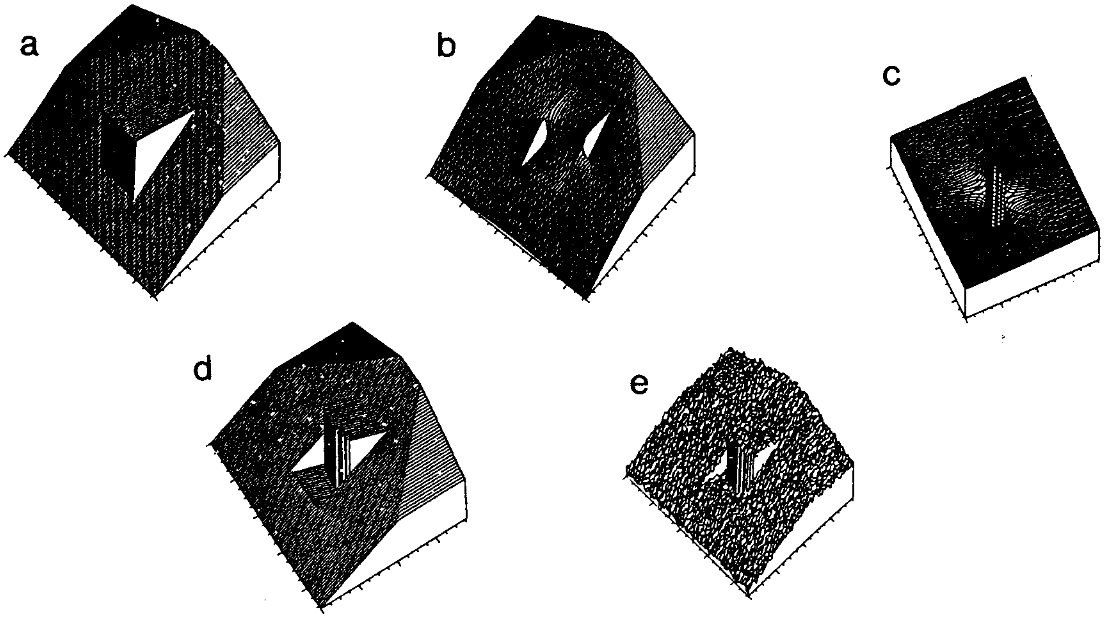

Fig.1 Example of 2D phase unwrapping. (a) Synthetic phase field with 2π phase step; irrotational (b) and rotational (c) phase field. The unwrapped phase (d) contains the ambiguous 2π step connecting two singular points. Even noisy phase ($\sigma_{noise} \simeq \pi/6$) has been unwrapped (e).

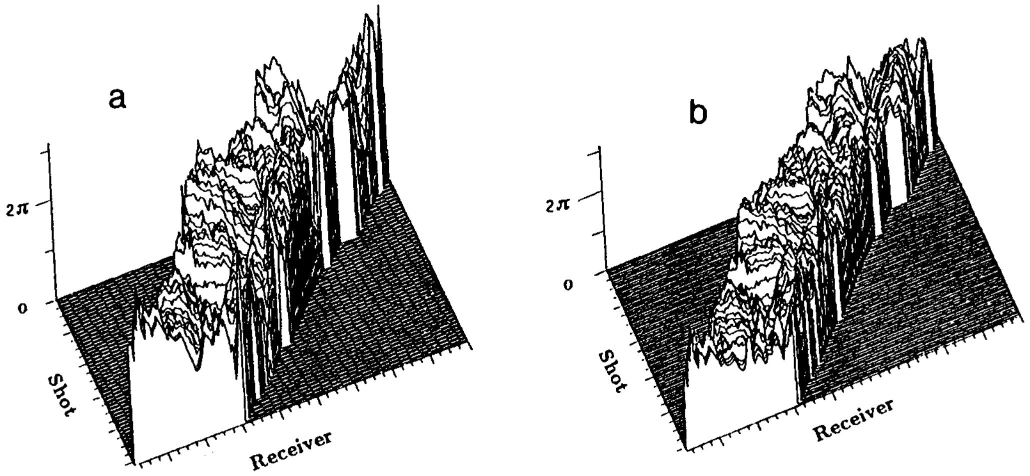

Fig.2 Unwrapped phase of the refracted first arrivals using line integration (a), phase field decomposition (b).